Ritz Method and Experimental Modal Analysis: CMS, CBR, AM, FIM, MLM and other acronyms

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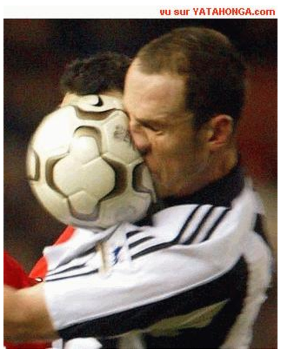

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Outline

- Introduction to Ritz Method
- Structural Modification
  - Theory
  - Structural Modification using M.P. Sensitivity
  - Example
- Sub-structuring
  - Abbreviated Theory
  - Example
- Conclusions

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Ritz Method

- The Ritz method is an approximate method for solving Partial Differential Equations.

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + \rho A\ddot{w} = f_z(x)$$

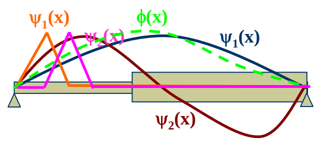
- Approximate *w(x,t)* with a set of basis functions $\psi_j(x)$

$$w(x,t) = \sum_{j=1}^{N} \psi_j(x) q_j(t)$$

- The PDE becomes an ODE

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = F \qquad (1)$$

- This is also known as the "Method of Assumed Modes," "Raleigh-Ritz Method," etc…
- Finite Element Method can be viewed as a special case.



$$\left[[K]\{x\} - \omega^2[M]\{\ddot{x}\}\right]\{p\} = 0$$

([K] and [M] are 2 by 2)

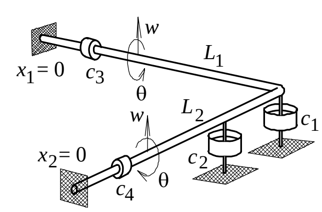
$$\begin{aligned}\{\phi(x)\} &= \{p\}^T\psi(x)\\ &= [0.93, 0.12]\psi(x)\\ &= 0.93\psi_1(x) + 0.12\psi_2(x)\end{aligned}$$

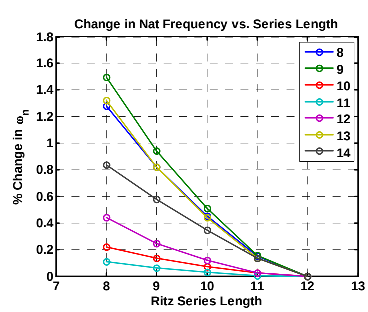
For a detailed discussion of the Ritz method, See:

Mechanical and Structural Vibrations, Ginsberg, 2001, Wiley.

Properties of the Ritz Method

- Natural Frequencies converge from above (Raleigh Ratio).
  - Example shows convergence for two cantilever beams joined at their free ends.
- Any set of linearly independent vectors can be used.
  - Finite Element Method: Use local Ritz vectors (piecewise linear, piecewise quadratic, etc…)
  - CMS: Component Mode Synthesis
  - CBR: Craig-Bampton Reduction
  - …
- These all seek to reduce the number of modes needed to get good results.
- One can experimentally measure only some of these types of modes:
  - Free modes
  - Mass loaded modes
  - Fixed interface modes (if the system is very flexible)





Applications of Ritz Method

- What can you do with the Ritz method and experimentally measured modes?
  - Just about anything you could do if you had approximate mass, stiffness and damping matrices!
  - Predict the effect of modifications to a structure
    - i.e. point mass, springs, dashpots
    - Could be used to correct experimentally measured modes for support (bungee) stiffness.
  - Join substructures and predict the modes and/or response of the master structure.
    - Combine component tests with FEA models for a master structure.
    - Add nonlinear forces between points and/or modes and simulate the response.
  - …

Structural Modification: Ritz Method

- Given a set of modal parameters, one has the following equations of motion for a structure:

$$[I]\{\ddot{\eta}\} + [\omega_r^2]\{\eta\} = [\Phi]^T\{F\}$$

$$\{y\} = [\Phi]\{\eta\}$$

- where {y} is a vector of physical coordinates, {η} are modal coordinates, $[\omega_r^{\;2}]$ is a diagonal matrix of natural frequencies squared and [Φ] is a matrix of mode vectors.
- Example: Equation of motion for a mass at point 'p'.

$$m_p \ddot{y}_p = F_p$$

- Combine:

$$\begin{bmatrix} m_p & 0 \\ 0 & [I] \end{bmatrix} \begin{Bmatrix} \ddot{y}_p \\ \ddot{\eta} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & [\omega_r^2] \end{bmatrix} \begin{Bmatrix} y_p \\ \{\eta\} \end{Bmatrix} = \begin{Bmatrix} F_p \\ [\Phi]^T\{F\} \end{Bmatrix}$$

Structural Modification: Ritz Method (2)

$$\begin{bmatrix} m_p & 0 \\ 0 & [I] \end{bmatrix} \begin{Bmatrix} \ddot{y}_p \\ \{\ddot{\eta}\} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & [\omega_r^2] \end{bmatrix} \begin{Bmatrix} y_p \\ \{\eta\} \end{Bmatrix} = \begin{Bmatrix} F_p \\ [\Phi]^T\{F\} \end{Bmatrix}$$

- Constraint equation to connect mass to structure:

$$y_p = [\Phi_p]\{\eta\} \quad \Rightarrow \quad \begin{bmatrix} -1 & [\Phi_p] \end{bmatrix} \begin{Bmatrix} y_p \\ \eta \end{Bmatrix} = 0$$

- But, y_p is known if $\{\eta\}$ is known:

$$\begin{Bmatrix} y_p \\ \{\eta\} \end{Bmatrix} = \begin{bmatrix} [\Phi_p] \\ [I] \end{bmatrix} \{\eta\} \quad \Rightarrow \quad \begin{Bmatrix} y_p \\ \{\eta\} \end{Bmatrix} = [B]\{\eta\}$$

- Eliminate y_p to find unconstrained EOM:

$$[\hat{M}] = [B]^T \begin{bmatrix} m_p & 0 \\ 0 & [I] \end{bmatrix} [B]$$

$$[\hat{M}]\{\ddot{\eta}\} + [\hat{K}]\{\eta\} = [\Phi]^T\{F\} \qquad [\hat{K}] = [B]^T \begin{bmatrix} 0 & 0 \\ 0 & [\omega_r^2] \end{bmatrix} [B]$$

$$\{\hat{Q}\} = [B]^T \begin{Bmatrix} F_p \\ [\Phi]^T\{F\} \end{Bmatrix}$$

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Structural Modification: Ritz Method (3)

- All of this can be easily implemented in a function.
- "ritzsmod.m" on the 860Modal page does just this:

```
% Structural modification using a experimentally determined mode vectors as
% a Ritz vector basis set.
%
% Computes the modal parameters of the system modified by adding the masses
% and stiffnesses in the vectors dm and km to nodes ns_m and ns_k;
%
% [wn_mod,zt_mod,phi_mod,M_mod,C_mod,K_mod] = ritzsmod(wn,zt,phi,dm,ns_m,dk,ns_k);
%
% "ritzmod" requires the following necessary inputs:
%   wn - vector of natural frequencies for the un-modified structure.
%   zt - (optional) vector of modal damping ratios.
%   phi - matrix of mass normalized mode vectors where each column
%       corresponds to a natural frequency in wn.
%   dm - vector of masses to be added to node numbers listed in ns_m.
%   dk - vector of stiffnesses to be added to node numbers listed in ns_k.
%       Each spring is between the node listed and ground.
```


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Structural Modification: Sensitivity Method

- Derivative based sensitivity analysis is often used to estimate the effect of modifications to a structure. (Ewins 2000)

$$\frac{\partial}{\partial p}\left[[K]\{x\} - \omega^2[M]\{\ddot{x}\}\right]\{\phi\} = 0 \quad \Rightarrow \quad \frac{\partial \omega_r}{\partial p} = \frac{1}{2\omega_r}\{\phi_r\}^T\left(\frac{\partial[K]}{\partial p} - \omega_r^2\frac{\partial[M]}{\partial p}\right)\{\phi_r\}$$

- For a mass addition at a single point this reduces to:

$$\Delta\omega_r = -\frac{1}{2}\Delta m\,\omega_r\,(\phi_{rk})^2$$

- This has also been used to scale mode shapes derived from Natural Excitation Modal Analysis.
- The Ritz method with a single mode gives the following, which reduces to the sensitivity results for small $\Delta m(\phi_{rk})^2$: $\hat{\omega}_r = \omega_r\sqrt{\frac{1}{1+\Delta m(\phi_{rk})^2}} \quad \Rightarrow \quad \Delta\omega_r = -\frac{1}{2}\Delta m\,\omega_r(\phi_{rk})^2$

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Structural Modification: Example

- Steel fixture (4.6”x1”x1” bar), treated as a mass and inertia added to the end of a 1”x0.75”x12” steel beam.

- Simulated in Salinas using 40 beam elements.
  - Weight of Beam: 2.5 lb
  - Weight of Fixture: 1.3 lb
- “ritzsmod.m” used (via “ritzsmod_test_script_FE.m”) used to remove mass and inertia.

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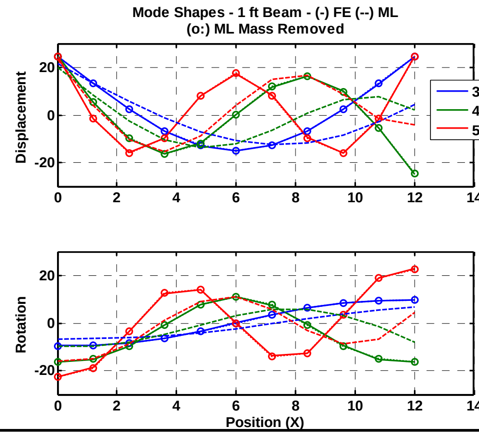
Structural Modification: Example (2)

Natural Frequencies (Hz) and Errors (%)						
Mode #	**ML (FE)**	**FF (FE)**	**FF (Ritz Method)**	**E (%) Ritz**	**A FF (Sensitivity)**	**E (%) Sensitivity**
3	703.766	1086.19	1085.32	*-0.08%*	835.54	*-23.1%*
4	1669.55	2988.13	2986	*-0.07%*	2055.7	*-31.2%*
5	3356.85	5845.21	5837.76	*-0.13%*	3673.9	*-37.1%*
6	6128.5	9639.3	17705.6	*83.68%*	6367	*-33.9%*

- Natural Frequencies and Mode Shapes from full (40 node) FE model compared with those estimated by "ritzsmod" using 6 modes (all modes below 6400 Hz).
- Errors in all but the 6th (last) mode are tiny.
- Mode shapes are dead on.
- Sensitivity estimate of natural frequencies has large errors (30%) because the mass addition is not small.



Joining Structures: Ritz Method (1)

- Consider the EOM for two independent structures, in the form we had previously for mass modification:

$$\begin{bmatrix} [I]_A & 0 \\ 0 & [I]_B \end{bmatrix} \begin{Bmatrix} \{\ddot{\eta}\}_A \\ \{\ddot{\eta}\}_B \end{Bmatrix} + \begin{bmatrix} [\omega_r^2]_A & 0 \\ 0 & [\omega_r^2]_B \end{bmatrix} \begin{Bmatrix} \{\eta\}_A \\ \{\eta\}_B \end{Bmatrix} = \begin{Bmatrix} [\Phi]_A^T\{F\} \\ [\Phi]_B^T\{F\} \end{Bmatrix}$$

- We connect them by enforcing some constraints:

$$(y_c)_A = (y_c)_B \quad \Rightarrow \quad [\Phi_c]_A\{\eta\}_A = [\Phi_c]_B\{\eta\}_B \quad \Rightarrow \quad [a]\begin{Bmatrix} \{\eta\}_A \\ \{\eta\}_B \end{Bmatrix}$$

- Eliminate some modal DOF using constraint equation to find unconstrained EOM:

$$[\widehat{M}]\{\ddot{\eta}\} + [\widehat{K}]\{\eta\} = [\Phi]^T\{F\}$$

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Joining Structures: Ritz Method (2)

- All of this can be easily implemented in a function.
- "ritzscomb.m" on the 860Modal page does just this:

```
% Structural modification using a experimentally determined mode vectors as
% a Ritz vector basis set.  Combines substructures (add or subtract)
% defined by modal parameters using the user supplied constraints and
% returns modal parameters for master structure.
%
% [wn_mod,zt_mod,phi_mod,subsys_ind,coord_ind,M_hat,C_hat,K_hat,phi_all] = ...
%       ritzscomb(subsys,carray,sgn_vec);
%
% INPUTS:
%   subsys - N_sub dimensional array with the following fields for each
%   substructure:  (i.e) subsys(1).wn = wn vector for subsystem #1.
%       wn - vector of natural frequencies for the un-modified structure.
%       zt - (optional) vector of modal damping ratios - set to zero
%           if not supplied.
%       phi - matrix of mass normalized mode vectors where each column
%           corresponds to a natural frequency in wn.
%       mmass - (optional) vector of modal masses - set to one if not
%           supplied.  (Set these to zero for residual flexibilities.)
%
%   carray - 3D array containing the constraints between subsystems.  Each
%       constraint is a 3-column matrix with the following columns:
%       carray(:,:,C_num) = [a1, pt1, subsys1;
%                            a2, pt2, subsys2;
%                            ...];
%       where a1 are the coefficients in an equation of the form:
%           a1*y1+a2*y2+...=0,
%       subsys1 is the index of the subsystem being joined (y1 is on
%           subsys1),
%       and pt1 is the index of the node in subsys1 that is being joined.
%       For Example:  To assign node 3 on subsystem 2 to have the same
%       displacement as node 4 on subsystem 1, the following matrix would
%       be used:  carray = [1, 3, 2; -1, 4, 1]
%           NOTE:  This assumes the coordinate systems are identical for
%           each component - use care if this is not the case!!
%
%   sgn_vec - [optional] vector of length N_sub of +1 or -1 values if each
%       substructure is being added or subtracted respectively to the
%       master system.  All substructures are added if this is not
%       supplied.
%
% OUTPUTS:
%   Modal params of combined system:  wn_mod,zt_mod,phi_mod
%   The rows in phi_mod correspond to a master set of coordinates
%   including each subsystem stacked sequentially.  Thus, the coordinates
%   that have been joined will be repeated in this vector.
%   subsys_ind - vector telling which subsystem each node belongs to.
%   coord_ind - vector telling which coordinate each node was in its
%       original subsystem.
%   M,C,K matrices for combined system in modal coordinates.
%   phi_all matrix to transfer from modal to physical coordinates
%
```

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Joining Structures: Example Problem

- Ritz Method used to combine models for 12" and 24" beams using modes for each substructure up to 6400 Hz .
  - Case 1: Free-Free (FF) modes – modal parameters of each FF beam used to predict combined system response:

  - Case 2: Mass-Loaded (ML) modes – parameters of modes with 4.6in end blocks used for each beam.

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Method

- Salinas models of 12” and 24” beams created with and without end blocks.
  - Each beam meshed with 0.3 inch long elements.
  - Solution of 12” and 24” beams joined was also found.
- The modal model for the case of Mass-Loaded modes was corrected using the “ritzsmod” script in order to remove the effects of the end masses. The modified modal models were then combined to form the 36-in beam.

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Correcting for End Mass and Inertia

	Natural Frequencies (Hz)							
Mode #	**A ML**	**A ML-corr**	**B ML**	**B ML-corr**	**A FF**	**B FF**	**E (%) ML**	**E (%) ML**
3	703.766	1085.32	213.845	271.945	1086.19	272.055	-0.08%	-0.04%
4	1669.55	2986	598.158	749.272	2988.13	749.554	-0.07%	-0.04%
5	3356.85	5837.76	1122.28	1468.17	5845.21	1468.62	-0.13%	-0.03%
6	6128.5	17705.6	1769.53	2425.59		2426.24		-0.03%
7			2646.01	3620.69		3622		-0.04%
8			3796	5053.16		5055.21		-0.04%
9			5205.86	13059.8				

- Chart compares Mass-Loaded natural frequencies with those obtained after analytically subtracting off the mass using “ritzsmod.m” with the Free-Free natural frequencies for systems A and B.
- The natural frequencies predicted match the FF natural frequencies remarkably well.

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Natural Frequencies for 36-in Beam

	Natural Frequencies (Hz)				
Mode #	FEA	FF	ML	E (%) FF	E(%) ML
3	121.0	128.0	121.0	5.8%	0.00%
4	333.3	364.0	333.3	9.2%	-0.02%
5	653.3	658.8	653.3	0.8%	0.00%
6	1079.7	1130.8	1079.7	4.7%	0.00%
7	1612.4	1781.8	1612.4	10.5%	0.00%
8	2251.3	2282.0	2252.7	1.4%	0.06%
9	2996.3	3176.3	3003.1	6.0%	0.23%
10	3847.2	4396.0	3855.6	14.3%	0.22%
11	4803.7	4934.3	4836.4	2.7%	0.68%

- Table shows natural frequencies of 36-in beam (1) predicted by FEA, (2) predicted using the Free-Free modes of the 12" and 24" beams and (3) predicted using Mass-Loaded modes of the 12" and 24" beams.
- Free-Free modes have errors ranging from 0% to 15%, while the Mass-Loaded modes show errors less than 1%.
- Error in natural frequency is largest for modes (of 36-in beam) that have large shear (derivative of rotation) at the connection point. (see mode shapes on next slide)

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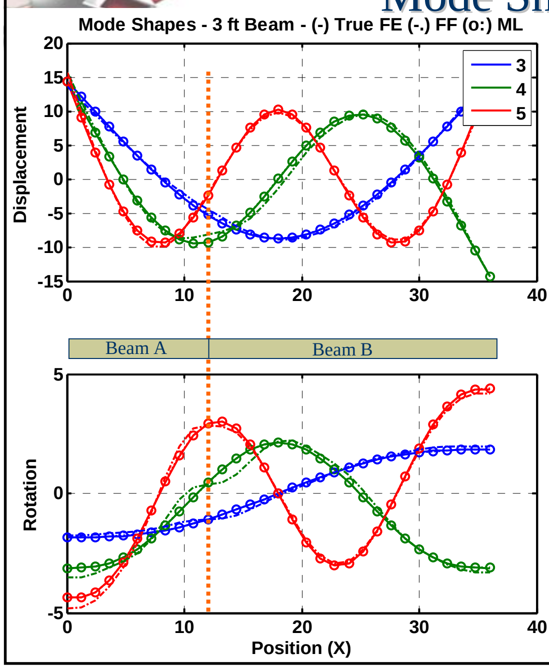
Mode Shapes (3-5)



- Mass loaded mode vectors match the analytical mode vectors more closely, especially the rotation component at the connection.
- Note: shear and moment in beam proportional to d^2u/dx^2 and d^3u/dx^3.
  - Free-Free modes all have zero shear and moment forces at the free end, so they cannot represent the combined structure well near the connection.

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Conclusions

- A set of modal parameters defines a set of equations of motion for a structure that can be used in much the same way that an FE or lumped parameter model would be.
  - Models can be joined or modified easily using canned programs.
  - An example illustrated that the Ritz method was much more accurate than sensitivity based methods for a free-free beam when the modification was substantial.
- The results obtained are best understood using Ritz theory.
  - Example illustrated that mass loaded modes gave much more accurate predictions of combined system modes than free-free modes.

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Future Work

- Compare this method with the FRF-based admittance approach.
- Investigate the sensitivity of these methods to measurement errors.
- Look at what happens when more complicated, multi-point connections are used.

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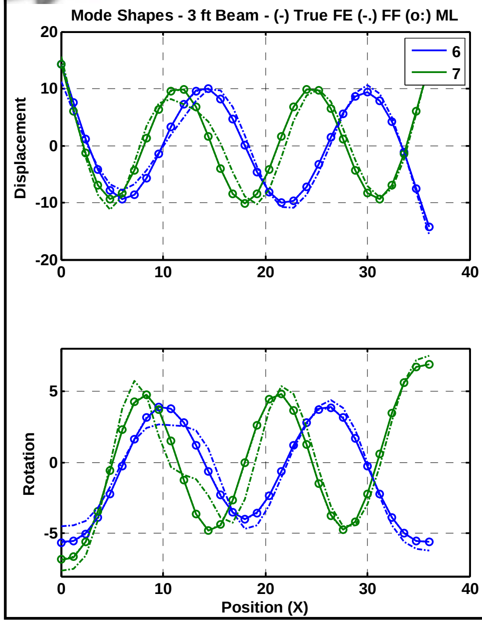
Mode Shapes (6-7)



- More mode shapes for the 36” beam.

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